Multiplier Ideals and Boundedness of Pluricanonical Maps

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Outline

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2. Informal definition of multiplier ideals
3. Actual definition of multiplier ideals
4. $\mathscr{I}^1 \leftrightarrow$ Sections which lift to Y
5. Lifting log pluricanonical forms

Motivation

Goal of the talk

- Not to present proofs of various theorems that were used.
- Rather, will introduce the ingredients that go into these theorems.
- Will try to give an idea of why we expect to have such theorems.

Thus, I will be imprecise in most places. Please check the paper for the correct statements.

What we want to do

Aim

Want to lift log pluricanonical forms.

- $X \subset Y$ smooth hypersurface
- $(X, \Delta) \subset (Y, \Gamma)$ with $(K_Y + \Gamma)|_X = K_X + \Delta$.

Loosely speaking, want to lift sections of $m(K_X + \Delta)$ to sections of $m(K_Y + \Gamma)$ for $m > 0$.

Remark

Have already done something similar before!

Siu's deformation invariance of plurigenera

Theorem (Siu)

$\pi : X \to T$ be a smooth projective family of general type varieties. Then:

$$h^0(mK_{X_t}) \text{ are constant for all } t \in T$$

Proof of Siu's theorem

1. Can assume T is a smooth affine curve. Suffices to show that every section of mK_{X_0} can be lifted to X.
2. Have asymptotic multiplier ideal $\mathcal{J}(\|mK_{X_0}\|)$.
   Sections of mK_{X_0} vanish along $\mathcal{J}(\|mK_{X_0}\|)$.
3. Define new multiplier ideal $\mathcal{J}(\|mK\|_0)$.
   Sections of mK_{X_0} which vanish along $\mathcal{J}(\|mK\|_0)$ lift to X.
4. Need appropriate containment relations between the two types of multiplier ideals!
   We need $\mathcal{J}(\|mK_{X_0}\|) \subset \mathcal{J}(\|mK\|_0)$. Unfortunately, this is not true.

5. Instead prove that we can find an effective divisor D s.t.:

$$\mathcal{J}(\|mK_{X_0}\|)(-D) \subset \mathcal{J}(\|mK\|_0) \text{ for all } m \geq 1$$

6. $\therefore$ The two multiplier ideals are asymptotically the same (i.e. as $m \to \infty$).
7. Sufficient to lift sections!

Strategy of our proof

We'll define two types of (asymptotic) multiplier ideals:

- $\mathscr{I}^0$ (corresponds to sections of $m(K_X + \Delta)$).
- $\mathscr{I}^1$ (corresponds to sections which lift to Y).

We then prove containment relations between the two multiplier ideals to get what we want.

Motivation for the definition of multiplier ideals

Remark

Siu's theorem is obvious if K_{X_0} is big and nef as we can use KV vanishing.

In case it's not nef, what do we do?
Try to extract a 'maximal' nef part from it!

Idea

Given $m > 0$, can find a birational map $\mu_m : Z_m \to X_0$ s.t.:

$$\mu_m^*(mK_{X_0}) = P_m + M_m$$
$$\text{(free part} + \text{fixed part)}$$

If there is one μ which serves as the μ_m for all m, then can take $P = \sup_m \frac{P_m}{m}$ as the desired nef part.
Get vanishing results for P.

Problem with the idea and how we fix it

Unfortunately, we don't have such a μ.

Fix

- Instead, consider new multiplier ideals associated to the positive part P_m of mK_{X_0}.
- Take the union (in X) over all m to get a new asymptotic multiplier ideal on X_0.
- Get vanishing results for these new ideals.

Informal definition of multiplier ideals

Old multiplier ideals

Definition

$X =$ smooth variety, $D =$ divisor.
$\mu : W \to X$ log resolution of D.
Then the multiplier ideal of D is defined as:

$$\mathcal{J}_D = \mu_*(\mathcal{O}_W(K_{W/X} - \lfloor \mu^* D \rfloor))$$

More generally, if we have a pair (X, Δ), we can define the multiplier ideal:

$$\mathcal{J}_{\Delta,D} := \mu_*(\mathcal{O}_W(K_{W/X} - \lfloor \mu^* \Delta \rfloor - \lfloor \mu^* D \rfloor))$$

New multiplier ideals

Definition

X = smooth variety, D = divisor.
$\mu : W \to X$ birational map such that $\mu^* D = P + M$; where P is 'free', M is 'fixed' (with everything being snc).
Then, define the multiplier ideal:

$$\mathcal{J}_M := \mu_*(\mathcal{O}_W(K_{W/X} - \lfloor M \rfloor))$$

This is the same as the **'adjoint ideal'** of D (cf. Section 9.3.E of Lazarsfeld's Positivity II).

New multiplier ideals for a pair

Definition

$(X, \Delta) =$ log smooth pair.
$\Delta =$ Reduced effective snc divisor.
$\mu : W \to X$ birational map such that $\mu^* D = P + M$; where P is 'free', M is 'fixed' (with everything being snc).
Set $\Theta :=$ Union of all divisors with discrepancy $= -1$.
Then, define the multiplier ideal:

$$\mathcal{J}_{\Delta,M} := \mu_*(\mathcal{O}_W(K_{W/X} + \Theta - \mu^*\Delta - \lfloor M \rfloor))$$

Reason for defining Θ

Set $E := K_{W/X} + \Theta - \mu^*\Delta$.
$\therefore E =$ those divisors with discrepancy ≥ 0.
In particular, E is effective exceptional and so:

$$\mathcal{J}_{\Delta,M} := \mu_*(\mathcal{O}_W(E - \lfloor M \rfloor))$$

is actually an ideal sheaf.

Actual definition of multiplier ideals

Setup

We'll do everything in the relative setting. So the notation will get slightly messy.

1. (Y, Γ) is a log smooth pair, $\Gamma =$ Reduced effective snc divisor.
2. $X \subset Y$ is a smooth hypersurface.
3. (X, Δ) is a log smooth pair, $\Delta =$ Reduced effective snc divisor s.t.:

$$(K_Y + \Gamma)|_X = K_X + \Delta$$

4. $\pi : Y \to S$ is a projective morphism.

Technical definitions

Definition

$D =$ Divisor on X.
Say that D is π-transverse for (X, Δ) if $\mathcal{O}_X(D)$ is π-generated at the generic point of every lc center of $K_X + \Delta$.
Define π-$\mathbb{Q}$-transverse in the natural way.

If π is the map to a point, then D being transverse just means that D (generically) avoids all the k-fold intersections of the components of Δ.

Definition

Say that a proper birational map $\mu : W \to X$ is **canonical** if every exceptional divisor extracted by μ has log discrepancy at least 1.

In this case, Θ is actually the strict transform of Δ.

Formal definition of (asymptotic) multiplier ideals

Definition

$D = \pi$-$\mathbb{Q}$-transverse divisor for (X, Δ).
For each $m > 0$, we 'resolve' the linear system $|mD|$ i.e.
Pick a canonical map $\mu_m : W_m \to X$ s.t. $\mu_m^*(mD) = P_m + M_m$ and:

1. P_m is $\pi \circ \mu_m$-free
2. Sections of mD are same as that of the sections of P_m i.e.:

$$(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(P_m) \to \pi_*(\mathcal{O}_X(mD))$$

   is an isomorphism.
3. M_m is effective. Everything in sight is snc.

Define $\mathscr{I}^0_{\Delta,D} := \bigcup_m \mathcal{J}_{\Delta, \frac{1}{m} M_m}$

Second type of multiplier ideals

Definition

$D = \pi$-$\mathbb{Q}$-transverse divisor for (Y, Γ).
We can make a very similar definition by asking for a birational map $\mu_m : W_m \to Y$ along with a decomposition:

$$\mu_m^*(mD) = Q_m + N_m$$

with Q_m being the 'free' part.
Define:

$$\mathscr{I}^1_{\Delta,D} := \bigcup_m \mathcal{J}_{\Delta,\frac{1}{m}N_m|_{X_m}}$$

where X_m =Strict transform of X.

This is again an ideal sheaf on X.

Well definedness

Theorem

The sheaves $\mathscr{I}^i_{\Delta,D}$ are well defined, i.e. they do not depend on the choice of μ_m.

Properties of new asymptotic multiplier ideals

$D = \pi$-$\mathbb{Q}$-transverse divisor for (Y, Γ).

1. $\mathscr{I}^1_{\Delta,D} \subset \mathscr{I}^0_{\Delta,D|_X}$.
2. There is an $m > 0$ which calculates the multiplier ideals i.e.:

$$\mathcal{J}^0_{\Delta,D} = \mathcal{J}_{\Delta,\frac{1}{m}M_m}$$
$$\mathcal{J}^1_{\Delta,D} = \mathcal{J}_{\Delta,\frac{1}{m}N_m|_X}$$

3. $B =$ effective divisor, then:

$$\mathscr{I}^i_{\Delta,D}(-B) \subset \mathscr{I}^i_{\Delta,D+B}$$

4. For $\alpha \geq 1$, we have $\mathscr{I}^i_{\Delta,\alpha D} \subset \mathscr{I}^i_{\Delta,D}$. (Bigger divisors $\implies$ deeper ideals)
5. If L is a π-free divisor, then we have:

$$\mathscr{I}^i_{\Delta,D} \subset \mathscr{I}^i_{\Delta,D+L}$$

$\mathscr{I}^1 \leftrightarrow$ Sections which lift to Y

Theorem

Theorem

$\mathrm{Im}(\pi_*\mathcal{O}_Y(D) \to \pi_*\mathcal{O}_X(D)) \subset \pi_*\mathcal{J}^1_{\Delta,D}(D)$.
(Sections of $\mathcal{O}_X(D)$ which lift to the whole of Y vanish on $\mathcal{J}^1_{\Delta,D}$.)

Proof

Choose $m > 0$ which calculates $\mathscr{I}^1_{\Delta,D}$. Say $\mu_m^* D = Q_m + N_m$.
We have on W_m:

$$Q_m \leq \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \leq \mu_m^* D$$

Push this forward via $\pi \circ \mu_m$:

$$\pi_*\mathcal{O}_Y(D) \subset (\pi \circ \mu_m)_* \mathcal{O}_{W_m}(\mu_m^* D - \lfloor \frac{N_m}{m} \rfloor) \subset \pi_*\mathcal{O}_Y(D)$$

Proof (contd.)

Thus, $\pi_* \mathcal{O}_Y(D) = (\pi \circ \mu_m)_* \mathcal{O}_{W_m}(\mu_m^* D - \lfloor \frac{N_m}{m} \rfloor)$.

Now observe that the image of RHS in $\pi_* \mathcal{O}_X(D)$ is contained in $\pi_* \mathscr{I}^1_{\Delta,D}(D)$.
Thus, we're done!

Converse

Theorem

Under suitable hypothesis, we have the inclusion:

$$\pi_*\mathcal{J}^1_{\Delta,D}(D+K_X+\Delta) \subset \mathrm{Im}(\pi_*\mathcal{O}_Y(D+K_Y+\Gamma) \to \pi_*\mathcal{O}_X(D+K_X+\Delta))$$

(Sections of $D+K_X+\Delta$ which vanish on $\mathcal{J}^1_{\Delta,D}$ lift to the whole of Y.)

We require vanishing results in the proof of this statement.

Lifting log pluricanonical forms

Clarification

- I'll first state an incorrect version of the main technical theorem. This is just to make the statement more digestible.
- Later, I'll indicate the tiny correction we have to make.

Comparing $\mathscr{I}^0$ with $\mathscr{I}^1$ (Incorrect version)

Theorem (Incorrect version)

$H =$ sufficiently π-very ample divisor, $A = (\dim(X) + 1)H$.
Assume that $K_X + \Delta$ is π-$\mathbb{Q}$-pseudoeffective. Then we have:

$$\mathscr{I}^0_{m(K_X+\Delta)+H|_X} \subset \mathscr{I}^1_{m(K_Y+\Gamma)+H+A}$$

for all $m > 0$.

Thus, the theorem is saying that $\mathscr{I}^0$ is contained in $\mathscr{I}^1$ if you add this fixed positive divisor A.
In particular, this fixed A works for all $m > 0$.

Remark

Thus, asymptotically in m (i.e. as $m \to \infty$), $\mathscr{I}^0$ and $\mathscr{I}^1$ are the same

Why is this incorrect?

- Look at the second term: $\mathscr{I}^1_{m(K_Y+\Gamma)+H+A}$.
- For this to be defined, we need $m(K_Y + \Gamma) + H + A$ to be π-$\mathbb{Q}$-transverse to (Y, Γ) **for all** $m > 0$.
- But this might not be the case because $K_Y + \Gamma$ is itself not transverse to (Y, Γ).
- To fix this, we perturb by a general divisor C so that $K_Y + \Gamma + C$ is transverse to (Y, Γ).

Comparing $\mathscr{I}^0$ with $\mathscr{I}^1$ (Correct version)

The setup is the same as before.

Theorem (Theorem 3.16 in HM)

$C \in \mathrm{Div}(Y)$ *not containing* X *s.t.:*

$$K_Y + \Gamma + C \text{ is } \pi\text{-}\mathbb{Q}\text{-transverse for } (Y, \Gamma)$$

Then we have:

$$\mathscr{I}^0_{m(K_X+\Delta)+H|_X}(-mC) \subset \mathscr{I}^1_{m(K_Y+\Gamma+C)+H+A}$$

Lifting log pluricanonical forms

The setup is the same as before.

Theorem (Theorem 3.17 in HM)

For any $m > 0$, the image of the natural map:

$$\pi_*\mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A) \to \pi_*\mathcal{O}_X(m(K_X + \Delta + C) + H + A)$$

contains the image of $\pi_*\mathcal{O}_X(m(K_X + \Delta) + H)$.
(We can lift sections of $m(K_X + \Delta) + H$ to the whole of Y.)

Thus, even though we might not be able to lift sections of $m(K_X + \Delta)$, we can do so after a small perturbation by H.